Journal of Statistical Physics, Vol. 112, Nos. 1/2, July 2003 (© 2003)

Extremal Reversible Measures for the Exclusion Process

Paul Jung[1]

Received September 29, 2002; accepted February 11, 2003

The invariant measures $\mathscr{I}$ for the exclusion process have long been studied and a complete description is known in many cases. This paper gives characterizations of $\mathscr{I}$ for exclusion processes on $\mathbb{Z}$ with certain reversible transition kernels. Some examples for which $\mathscr{I}$ is given include all finite range kernels that are asymptotically equal to $p(x, x+1)=p(x, x-1)=1/2$. One tool used in the proofs gives a necessary and sufficient condition for reversible measures to be extremal in the set of invariant measures, which is an interesting result in its own right. One reason that this extremality is interesting is that it provides information concerning the domains of attraction for reversible measures.

KEY WORDS: Exclusion process; invariant measures; extremal invariant measures; domains of attraction.

1. INTRODUCTION

Given a countable set $\mathscr{S}$ and a corresponding probability transition function $p(x, y)$ satisfying $\sup_y \sum_x p(x, y) < \infty$, IPS (Liggett[8]) constructs and describes the exclusion process on $\{0, 1\}^{\mathscr{S}}$. Its generator is given by the closure of the operator Ω on $\mathscr{D}(\{0, 1\}^{\mathscr{S}})$, the set of all functions on $\{0, 1\}^{\mathscr{S}}$ that depend on finitely many coordinates. If $f \in \mathscr{D}(\{0, 1\}^{\mathscr{S}})$ and η_{xy} is defined as

$$\eta_{xy}(u) = \begin{cases} \eta(y) & \text{if} \quad u = x \\ \eta(x) & \text{if} \quad u = y \\ \eta(u) & \text{if} \quad u \neq x, y \end{cases}$$

[1] Department of Mathematics, University of California, Los Angeles, California 90095-1555; e-mail: pjung@math.ucla.edu

then

$$\Omega f(\eta) = \sum_{\eta(x)=1,\, \eta(y)=0} p(x, y)[f(\eta_{xy}) - f(\eta)].$$

The semigroup of this process will be denoted by $S(t)$.

When $p(x, y) = p(y, x)$ the process has been completely studied in that a full description of its invariant measures is known as well as their respective domains of attraction. The asymmetric exclusion process on the other hand has been much more elusive. General classes of invariant measures are known in the two cases where $p(x, y)$ is doubly stochastic (i.e., $\sum_{x \in \mathscr{S}} p(x, y) = 1$ for all $y \in \mathscr{S}$) or when there exists a reversible measure $\pi(x) > 0$ on $\mathscr{S}$ (i.e., a measure satisfying $\pi(x)\, p(x, y) = \pi(y)\, p(y, x)$). However, a complete description of $\mathscr{I}$ is known only in the three cases when either

(a) $p(x, y)$ is reversible and positive recurrent for either the particles or holes (1's or 0's);[7]

(b) $p(x, y)$ corresponds to certain random walks on $\mathbb{Z}$;[3,7] or

(c) $p(x, y)$ corresponds to a birth and death chain on $\mathbb{Z}^+$.[7]

Almost nothing is known about the domains of attraction concerning invariant measures in the asymmetric case, although we note here that there are some nice theorems concerning the case where $p(x, y)$ is an asymmetric simple random walk on $\mathbb{Z}$ (see Liggett[9]).

Our purpose in this paper is to shed some more light on the problem of classifying $\mathscr{I}$ and its respective domains of attraction for the asymmetric exclusion process when a reversible measure $\pi(x)$ exists for $p(x, y)$. In order to describe the results of this paper we must first discuss case (a) and state a special case of (b) above.

We start by stating what is known for the mean zero case of (b). Let ν_ρ be the product measure on $\{0, 1\}^{\mathscr{S}}$ with marginals $\nu_\rho\{\eta: \eta(x) = 1\} = \rho$. Liggett[7] uses a coupling of two exclusion processes to show that when $p(x, y) = p(0, y - x)$, $\sum_x |x|\, p(0, x) < \infty$, and $\sum_x x p(0, x) = 0$ on $\mathbb{Z}$ the set of extremal invariant measures is

$$\mathscr{I}_e = \{\nu_\rho : 0 \leqslant \rho \leqslant 1\}. \tag{1}$$

Before describing the invariant measures for case (a), we define some extremal reversible invariant measures $\{\nu^{(n)}\}$ when a reversible measure $\pi(x)$ satisfying

$$\sum_x \pi(x)/[1 + \pi(x)]^2 < \infty \tag{2}$$

exists. This family of extremal reversible measures was first discovered by Liggett. In particular, he breaks down (2) into three cases and writes

1. If $\sum_x \pi(x) < \infty$, let $A_n = \{\eta\colon \sum_x \eta_x = n\}$ for nonnegative integers n.

2. If $\sum_x 1/\pi(x) < \infty$, let $A_n = \{\eta\colon \sum_x [1-\eta_x] = n\}$ for nonnegative integers n.

3. If $\sum_x \pi(x)/[1+\pi(x)]^2 < \infty$, $\sum_x \pi(x) = \infty$, and $\sum_x 1/\pi(x) = \infty$, there exists a $T \subset S$ for which $\sum_{x \in T} \pi(x) < \infty$ and $\sum_{x \notin T} 1/\pi(x) < \infty$. In this case, let

$$A_n = \left\{\eta\colon \sum_{x \in T} \eta(x) - \sum_{x \notin T} [1-\eta(x)] = n\right\}$$

for integers n.

To define $\{\nu^{(n)}\}$, let ν^c be the product measure with marginals $\nu^c\{\eta\colon \eta(x) = 1\} = \frac{c\pi(x)}{1+c\pi(x)}$. Liggett shows that the measures

$$\nu^{(n)}(\cdot) = \nu^c(\cdot \mid A_n), \qquad \nu^{(\infty)}(\cdot) = \delta_1, \qquad \nu^{(-\infty)}(\cdot) = \delta_0$$

are the unique stationary distributions for the positive recurrent Markov chains on A_n. A simple consequence of Theorem B52 in ref. 9 is that the reversible measures $\{\nu^{(n)}\}$ are extremal in the set of invariant measures. For the first two cases in the trichotomy of (2) above, these are the only extremal invariant measures. These first two cases correspond exactly to (a) above. Note that changing T in the third case of the trichotomy above amounts to a relabeling of the sequence $\{\nu^{(n)}, n \in \mathbb{Z}\}$.

Whenever a reversible measure $\pi(x)$ on $\mathcal{S}$ exists, the product measures $\{\nu^c\}$ are well-defined. Theorem VIII.2.1 in IPS tells us that these measures are invariant for the exclusion process. Applying Kakutani's Dichotomy (e.g., p. 244 of ref. 4) we have that $\sum_x \pi(x)/[1+\pi(x)]^2 = \infty$ is a necessary and sufficient condition for the measures $\{\nu^c\colon 0 \leqslant c \leqslant \infty\}$ to be mutually singular. Since all the results in this paper concern the reversible measures $\{\nu^c\}$, we will assume throughout the rest of the paper that $\pi(x)$ satisfying $\pi(x)\, p(x, y) = \pi(y)\, p(y, x)$ exists.

In Section 2 we prove Theorem 2.1 which states that $\sum_x \pi(x)/[1+\pi(x)]^2 = \infty$ is exactly the situation in which the measures ν^c are extremal invariant. Not only does this result have some nice applications, but knowing that an invariant measure is extremal in the set of invariant measures has always been an interesting question concerning particle systems. Examples of such results are Theorem III.1.17 in ref. 9 and Theorem 1.4 in ref. 10. The main reason extremality of invariant measures is interesting is its close connection with ergodicity. This is seen by the application Theorem III.1.17 in ref. 9 to prove Theorem III.4.8 in ref. 9 concerning the tagged particle process; it is again seen by the application of Theorem 1.4 in ref. 10 to certain central limit theorems given in ref. 6. In particular, if the initial measure for a process is an extremal invariant measure then the process evolution is ergodic with respect to time shifts.

Sections 3 and 4 use Theorem 2.1 to extract information about the invariant measures of the process on $\mathbb{Z}$. In particular, Section 3 modifies Liggett's original proof of the result stated above Eq. (1) to obtain the following result:

Theorem 1.1. Let $\mathbb{Z}$ be irreducible with respect to a transition kernel $p(x, y)$ for which there exists a reversible measure $\pi(x)$. Suppose $q_i(z)$ is a transition kernel such that $\sum_z zq_i(z) = 0$ and $\sum_z |z|\, q_i(z) < \infty$ for $i = 1, 2$, and suppose that

$$\lim_{K\to\infty} \sum_{x\geqslant 0} \sum_{|z|\geqslant|x-K|} |p(x, x+z) - q_1(z)| = 0 \qquad \text{and}$$
$$\lim_{K\to\infty} \sum_{x\leqslant 0} \sum_{|z|\geqslant|x+K|} |p(x, x+z) - q_2(z)| = 0. \tag{3}$$

(a) If $\sum_x \pi(x)/[1+\pi(x)]^2 = \infty$ then $\mathscr{I}_e = \{\nu^c : 0 \leqslant c \leqslant \infty\}$.

(b) If $\sum_x \pi(x)/[1+\pi(x)]^2 < \infty$ then $\mathscr{I}_e = \{\nu^{(n)}\}$.

In essence the above theorem says that when the transition probabilities are asymptotically translation invariant and have an asymptotic mean of zero, the reversible measures are the only invariant measures. Theorem 1.1 is merely an extension (in the case where $\pi(x)$ exists) of the theorem proved by Liggett[7] which is stated above Eq. (1).

Condition (3) may seem somewhat daunting, but note that if $\lim_{x\to\infty} p(x, x+z) = q_1(z)$, $\lim_{x\to-\infty} p(x, x+z) = q_2(z)$, and $p(x, y)$ has finite range (i.e., $p(x, y) \equiv 0$ if $|x-y| > n$ for some n), then (3) and $\sum_z |z|\, q_1(z) < \infty$ are both automatically satisfied. Also, the below condition which is somewhat easier to grasp than (3) implies (3):

$$\sum_{x\geqslant 0} \sum_z |p(x, x+z) - q_1(z)| < \infty \quad \text{and} \quad \sum_{x\leqslant 0} \sum_z |p(x, x+z) - q_2(z)| < \infty.$$

A typical situation for which the theorem holds is when the transition rates are nearest-neighbor and are given by $p(x, x+1) = p(x, x-1) = 1/2$ except for finitely many x.

Note that the premises of the theorem together with the assumption that a reversible $\pi(x)$ exists imply that $q_i(z)$ must be symmetric. To see this suppose $q_1(z)$ is not symmetric. Also, assume that $q_1(z_1) > q_1(-z_1) > 0$ for some $z_1 \in \mathbb{N}$. We can do this without loss of generality since $q_1(z) > 0$ implies $q_1(-z) > 0$ by the reversibility of $\pi(x)$. The mean zero assumption tells us there exists $z_2 \in \mathbb{N}$ such that $q_1(z_2) < q_1(-z_2)$. If z_3 is a multiple of both z_1 and z_2 then since $p(x, x+z) \to q_1(z)$ we can find x_1 so that for $x > x_1$, $\pi(x) < \pi(x+z_3)$. But we can also find x_2 so that for $x > x_2$, $\pi(x) > \pi(x+z_3)$, a contradiction. So $q_1(z)$ must be symmetric. The proof that $q_2(z)$ is symmetric follows similarly.

The proof of the above theorem follows Liggett's original outline and does not actually require Theorem 2.1. However, the usefulness of Theorem 2.1 is seen in the simplification of one part of Liggett's original proof.

In Section 4 we prove a theorem concerning the nearest-neighbor exclusion process on $\mathbb{Z}$. For the statement of the theorem we will need the following definitions.

Let $\mathscr{L}^-$ be the set of limit points of $\{\pi(x), x < 0\}$ and $\mathscr{L}^+$ be the set of limit points of $\{\pi(x), x > 0\}$.

Theorem 1.2. Suppose that $\inf_{|x-y|=1} p(x, y) > 0$ for a nearest-neighbor exclusion process on $\mathbb{Z}$. Then nonreversible invariant measures can exist only when either (a) $\mathscr{L}^- = \{0\}$ and $\mathscr{L}^+ = \{\infty\}$ or (b) $\mathscr{L}^- = \{\infty\}$ and $\mathscr{L}^+ = \{0\}$.

The above theorem in no way guarantees the existence of nonreversible invariant measures as seen by the following example. Let

$$p(-1, -2) = p(-1, 0) = p(0, -1) = p(0, 1) = 1/2,$$
$$p(x, x+1) = 1 - p(x, x-1) = \frac{|x|+1}{2\,|x|} \quad \text{otherwise.} \tag{4}$$

This transition gives us situation (a) in the theorem above. The reversible invariant measures $\{\nu^c\}$ certainly exist, but it is easy to see that condition (b) of Theorem 1.1 is satisfied by (4), therefore there are no nonreversible invariant measures.

A curious aside is as follows. If in this example we start this process off with initial measure ν_ρ and take the limit of some converging sequence of measures

$$\frac{1}{T_n}\int_0^{T_n} \nu_\rho S(t)\,dt \tag{5}$$

then Theorem I.1.8 in IPS says that this limit is an invariant measure for the process. In view of the previous discussion, this limit must converge to some mixture of the extremal invariant measures $\{\nu^{(n)}, -\infty \leqslant n \leqslant \infty\}$. It would be interesting indeed to find out which mixture (5) converges to. Note here that we started off with an initial state that concentrates on an uncountable number of states, but the limiting distribution concentrates on a countable number of states (which may very well be just the point masses of all 0's and all 1's).

If $p(x, y)$ is an asymmetric, nearest-neighbor random walk kernel with nonzero mean then we have one of the situations described in the theorem above, and one might correctly guess that there exists some nonreversible invariant measure. In fact, a well-known result of Liggett[7] proves that the measures

$$\{\nu_\rho : 0 \leqslant \rho \leqslant 1\} \tag{6}$$

are invariant measures. Since any limit of (5) is invariant, we intuitively might have expected this. More precisely, if there were no nonreversible measures then this limit would presumably be a mixture of the reversible measures ν^c. But it is intuitive that there is no mixture of ν^c's to which this limit could converge, leading us to believe that the limit converges to some other measure.

We note here that the set of measures in (6) is the same as the set of measures in (1) but are of an entirely different nature. In the setting of (1) the measures $\{\nu_\rho : 0 \leqslant \rho \leqslant 1\}$ are reversible and constitute the entire set of extremal invariant measures. On the other hand, under the current setting, the measures $\{\nu_\rho : 0 \leqslant \rho \leqslant 1\}$ are not reversible and

$$\mathscr{I}_e = \{\nu_\rho : 0 \leqslant \rho \leqslant 1\} \cup \{\nu^c : 0 \leqslant c \leqslant \infty\}.$$

The discussion in the previous paragraphs might make us wonder for which transition kernels a nonreversible invariant measure exists. To gain more insight into the situation we introduce a concept known as the *flux* of an invariant measure μ. We will continue to assume that the transition probabilities are nearest-neighbor, but we will no longer assume they are translation invariant. Define

$$\text{flux}(\mu) = p(x, x+1)\, \mu\{\eta: \eta(x) = 1, \eta(x+1) = 0\} \\ - p(x+1, x)\, \mu\{\eta: \eta(x) = 0, \eta(x+1) = 1\}. \qquad (7)$$

Let $1_x(\eta) = \eta(x)$ be the indicator function of $\{\eta(x) = 1\}$. By computing the positive and negative terms of the left-hand side of $\int \Omega 1_x \, d\mu = 0$ it can be seen that flux(μ) is independent of x.

When an invariant measure μ is reversible it can easily be seen from (7) that flux$(\mu) = 0$. So if an invariant measure exists whose flux is nonzero it must be nonreversible. For the process with $p(x, x+1) > 1/2$ and $p(x, x-1) = 1 - p(x, x+1)$, the invariant measures $\{\nu_\rho: 0 \leqslant \rho \leqslant 1\}$ all have a positive flux with the flux being maximized when $\rho = 1/2$ (a full discussion of this can be found in either ref. 5 or Part III of ref. 9). This positive flux is the reason why (6) is fundamentally different from (1). It would be quite nice if one could prove that some nonreversible invariant measure exists whenever $p(x, x+1) > 1/2 + \epsilon$ for all x. The ϵ here serves the role of providing some positive flux in the limit.

Finally, Section 5 will apply Theorem 2.1 to give information concerning the domains of attraction (in the Cesaro sense) of reversible measures in the case where $\sum_x \pi(x)/[1+\pi(x)]^2 = \infty$. The results of Section 5 only give sufficient conditions for Cesaro convergence to an invariant measure, but are nonetheless interesting since so little is known concerning domains of attraction for the asymmetric exclusion process. The key known results concerning domains of attraction of asymmetric exclusion processes are stated in Andjel *et al.*[2] They concern the limiting distribution of exclusion processes with asymmetric nearest-neighbor random walk kernels when the initial measures are certain product measures. To get an idea of how difficult it is to prove anything of this sort, we refer the reader to ref. 2.

The fact that Theorem 5.1 concerns Cesaro convergence rather than the usual weak convergence, while undesirable, is not so bad since many results in particle systems concern Cesaro convergence (see Section I.1 in IPS). One notable example of this is the main result of Andjel[1] which concerns the Cesaro convergence of certain initial product measures when the transition kernel of the exclusion process is an asymmetric nearest-neighbor random walk. In fact these results were later shown to be true for weak convergence (this was the goal of Andjel *et al.*[2]). We note here that Theorem 5.1 does not use the property of reversibility, therefore one can apply the theorem to situations in which one knows that a particular invariant measure is extremal in the set of invariant measures.

2. EXTREMAL REVERSIBLE MEASURES

In this section we state and prove Theorem 2.1. The common technique used in the proof of this theorem and in the proofs of most of the other results in this paper is the coupling technique. We now define the basic coupling of η_t and ξ_t which lets the two exclusion processes move together as much as possible. The generator for this coupling is the closure of the operator $\tilde{\Omega}$ defined on $\mathscr{D}(\{0, 1\}^{\mathscr{S}} \times \{0, 1\}^{\mathscr{S}})$:

$$\begin{aligned}\tilde{\Omega} f(\eta, \xi) = & \sum_{\eta(x)=\xi(x)=1,\, \eta(y)=\xi(y)=0} p(x, y)[f(\eta_{xy}, \xi_{xy}) - f(\eta, \xi)] \\ & + \sum_{\eta(x)=1,\, \eta(y)=0 \text{ and } (\xi(y)=1 \text{ or } \xi(x)=0)} p(x, y)[f(\eta_{xy}, \xi) - f(\eta, \xi)] \\ & + \sum_{\xi(x)=1,\, \xi(y)=0 \text{ and } (\eta(y)=1 \text{ or } \eta(x)=0)} p(x, y)[f(\eta, \xi_{xy}) - f(\eta, \xi)].\end{aligned}$$

Theorem 2.1. Suppose $\mathscr{S}$ is irreducible with respect to $p(x, y)$. Then the measures ν^c are extremal invariant if and only if $\sum_x \pi(x)/[1+\pi(x)]^2 = \infty$.

Proof. The discussion on p. 383 of IPS shows that if $\sum_x \pi(x)/[1+\pi(x)]^2 < \infty$ then the measures ν^c are not extremal invariant giving us one direction of the theorem. We will prove the other direction.

Assume throughout that $0 < c < \infty$. Since the measures ν^c are invariant and since all bounded continuous functions can be approximated uniformly by functions that depend on finitely many coordinates then by Theorem B52 in ref. 9, we need only show that for any two functions f and g which depend on finitely many coordinates

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T E^{\nu^c} f(\eta_0)\, g(\eta_t)\, dt = \int f\, d\nu^c \int g\, d\nu^c.$$

We claim that to show the above equation holds, it is enough to show that for any finite $A \subset \mathscr{S}$ and for $\mu^c_{1, A}(\cdot) = \nu^c(\cdot \mid \{\eta: \eta(x) = 1\ \forall x \in A\})$

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \mu^c_{1, A} S(t)\, dt = \nu^c. \tag{8}$$

To see this define the measures $\mu^c_{\zeta, A}(\cdot) = \nu^c(\cdot \mid \{\eta(x) = \zeta(x)\ \forall x \in A\})$ where ζ is a configuration on $\{0, 1\}^A$. We can write the measure ν^c as a linear combination

$$\nu^c = \sum_{\zeta \in \{0, 1\}^A} a_\zeta \mu^c_{\zeta, A}$$

where we use the convention that $\zeta = i$ is the configuration in $\{0, 1\}^A$ such that $\zeta(x) = i$ for all $x \in A$. For

$$f_A = \begin{cases} 1 & \text{when } \eta(x) = 1 \text{ for all } x \text{ in the finite set } A \\ 0 & \text{otherwise} \end{cases}$$

we have that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T E^{\nu^c} f_A(\eta_0)\, g(\eta_t)\, dt = \lim_{T \to \infty} \frac{1}{T} \int_0^T a_1 \int S(t)\, g(\eta)\, d\mu_{1,A}^c\, dt$$

$$= \int f_A\, d\nu^c \int g\, d\nu^c$$

which proves the claim.

Define $\mu_{0,A}^c$ similarly to the way we defined $\mu_{1,A}^c$. If we assume a fixed A then we can drop the subscript A so as to write $\mu_i^c = \mu_{i,A}^c$. The rest of the proof will now argue that (8) holds.

Choose $\delta > 0$ and couple the processes η_t and ξ_t using the basic coupling starting with measures μ_0^c and μ_1^c so that η_0 and ξ_0 disagree only for $x \in A$. In particular, since the basic coupling is the coupling which allows η_t and ξ_t to move together as much as possible, then η_t and ξ_t can differ at most at n sites where $|A| = n$.

If there exists $\bar{T}$ such that for all $T > \bar{T}$

$$\frac{1}{T} \int_0^T [\mu_1^c S(t)\{\xi(0) = 1\} - \mu_0^c S(t)\{\eta(0) = 1\}]\, dt \leqslant \delta$$

then we must have that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \mu_1^c S(t)\{\xi(0) = 1\}\, dt = \lim_{T \to \infty} \frac{1}{T} \int_0^T \mu_0^c S(t)\{\eta(0) = 1\}\, dt.$$

Keeping in mind the way that η_t and ξ_t are coupled, irreducibility then tells us that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \mu_1^c S(t)\, dt = \lim_{T \to \infty} \frac{1}{T} \int_0^T \mu_0^c S(t)\, dt.$$

But the measure ν^c lies stochastically between the left-hand side and the right-hand side of the equation above, so in fact we must have that (8) holds.

We can therefore assume to the contrary that there exists a $\delta > 0$ and a sequence $\{T_n\}$ such that

$$\frac{1}{T_n} \int_0^{T_n} [\mu_1^c S(t)\{\xi(0)=1\} - \mu_0^c S(t)\{\eta(0)=1\}]\, dt > \delta \tag{9}$$

for all n.

Pick $\epsilon > 0$ so that

$$\nu^{c+\epsilon}\{\xi(0)=1\} - \nu^{c-\epsilon}\{\eta(0)=1\} < \delta/3.$$

Using the basic coupling once more, couple the processes η_t and ξ_t starting off in the measures μ_1^c and $\nu^{c+\epsilon}$ so that $\lambda_1\{(\eta, \xi): \eta(x) \leqslant \xi(x)$ for all $x \in \mathscr{S} \setminus A\} = 1$ where λ_1 is the coupling measure. If $\hat{\mu}^c = \nu^c(\cdot \mid \{\eta: \eta(x) = 0$ for some $x \in A\})$ then

$$\nu^c = \gamma \mu_1^c + (1-\gamma)\, \hat{\mu}^c$$

for $\gamma = \nu^c\{\eta: \eta(x) = 1\ \forall x \in A\}$. Couple the measures $\hat{\mu}^c$ and $\nu^{c+\epsilon}$ in a way similar to λ_1 so that we get another coupling measure λ_2.

Choose a subsequence $\{T_{n_k}\}$ so that we can define some limiting invariant measure

$$\omega_1 = \lim_{k \to \infty} \frac{1}{T_{n_k}} \int_0^{T_{n_k}} \lambda_1 S(t)\, dt.$$

Let ν_1^c be the η-marginal measure of ω_1 so that in particular

$$\nu_1^c = \lim_{k \to \infty} \frac{1}{T_{n_k}} \int_0^{T_{n_k}} \mu_1^c S(t)\, dt.$$

To complete the proof of the theorem we will need the following lemma:

Lemma 2.2. $\nu^{c+\epsilon} \geqslant \nu_1^c$.

Proof of Lemma 2.2. Let $f_x(\eta, \xi) = [1-\eta(x)]\, \xi(x)$, $D_m = \{(\eta, \xi)$: $\eta(x) > \xi(x)$ at exactly m sites$\}$, and $D = \bigcup_{m \geqslant 1} D_m$. If $\nu^{c+\epsilon} \not\geqslant \nu_1^c$ then it must be that $\omega_1(D) > 0$. We claim that this implies

$$\int_D \sum_x f_x\, d\omega_1 = 0.$$

To prove the claim, assume to the contrary that $\int_D \sum_x f_x \, d\omega_1 > 0$ so that there exist sites for which $\eta(x) < \xi(x)$. Let M be the largest m for which $\omega_1(D_m) > 0$. Then by the irreducibility condition and by the fact that there exist sites for which $\eta(x) < \xi(x)$ we have $\omega_1 S(t)(D_M) < \omega_1(D_M)$ for $t > 0$. But this is a contradiction to the invariance of ω_1 proving the claim.

Now if the two processes η_t and ξ_t have the measures ν^c and $\nu^{c+\epsilon}$ respectively then let ω be the coupling measure for $\{(\eta_t, \xi_t)\}$ which concentrates on $\nu^c \leqslant \nu^{c+\epsilon}$. For this coupling, the ω probability that $f_x(\eta, \xi) = 1$ for a given x is equal to the left-hand side below:

$$\frac{(c+\epsilon)\,\pi(x)}{1+(c+\epsilon)\,\pi(x)} - \frac{c\pi(x)}{1+c\pi(x)} > \frac{\epsilon\pi(x)}{[1+(c+\epsilon)\,\pi(x)]^2}.$$

Since $\sum_x \pi(x)/[1+\pi(x)]^2 = \infty$, by the Borel–Cantelli Lemma the ω probability that $\sum_x f_x = \infty$ is equal to 1. The measure ω_1 is absolutely continuous with respect to ω since

$$\omega = \gamma\lambda_1 + (1-\gamma)\,\lambda_2 = \gamma\omega_1 + (1-\gamma) \lim_{k\to\infty} \frac{1}{T_{n_k}} \int_0^{T_{n_k}} \lambda_2 S(t)\, dt$$

where λ_2 is as defined above. Therefore $\int_E \sum_x f_x \, d\omega_1 = \infty$ for any set E with positive ω_1 measure which contradicts $\int_D \sum_x f_x \, d\omega_1 = 0$ so it must be that $\omega_1(D) = 0$ proving the lemma. ▮

We now turn back to the proof of the theorem. Since by the lemma we have $\nu^{c+\epsilon} \geqslant \nu_1^c$, then there exists a K such that for all $k > K$

$$\frac{1}{T_{n_k}} \int_0^{T_{n_k}} \mu_1^c S(t)\{\eta(0) = 1\}\, dt - \nu^{c+\epsilon}\{\xi(0) = 1\} < \delta/3.$$

If ν_0^c is some limiting measure of

$$\frac{1}{T_{n_{k_l}}} \int_0^{T_{n_{k_l}}} \mu_0^c S(t)\, dt$$

then an argument similar to that used in Lemma 2.2 shows that $\nu^{c-\epsilon} \leqslant \nu_0^c$. There then exists an L such that for $l > L$

$$\nu^{c-\epsilon}\{\eta(0) = 1\} - \frac{1}{T_{n_{k_l}}} \int_0^{T_{n_{k_l}}} \mu_0^c S(t)\{\xi(0) = 1\}\, dt < \delta/3.$$

Altogether we have for $l > L$,

$$\frac{1}{T_{n_{k_l}}} \int_0^{T_{n_{k_l}}} [\mu_1^c S(t)\{\xi(0) = 1\} - \mu_0^c S(t)\{\eta(0) = 1\}]\, dt < \delta$$

which contradicts inequality (9) so it must be that (8) holds completing the proof of the theorem. ▮

3. THE ASYMPTOTICALLY MEAN ZERO PROCESS ON $\mathbb{Z}$

In this section we prove Theorem 1.1. To do so we will need to define $\tilde{\mathscr{I}}$ as the set of invariant measures for the basic coupling and $\tilde{\mathscr{I}}_e$ as its extreme points.

Recall that $f_x(\eta, \xi) = [1-\eta(x)]\, \xi(x)$. In order to simplify the notation we further define the functions

$$h_{yx}(\eta, \xi) = [1-\eta(y)][1-\xi(y)]\, f_x(\eta, \xi),$$

$$g_{yx}(\eta, \xi) = \eta(y)\, \xi(y)\, f_x(\eta, \xi), \qquad \text{and}$$

$$f_{yx}(\eta, \xi) = \eta(y)[1-\xi(y)]\, f_x(\eta, \xi).$$

In particular, for T a finite subset of $\mathscr{S}$ we have

$$\begin{aligned}\tilde{\Omega}\left(\sum_{x\in T} f_x(\eta, \xi)\right) = &- \sum_{x\in T,\, y\in \mathscr{S}} (p(x, y)+p(y, x))\, f_{yx}(\eta, \xi)\\ &+ \sum_{x\in T,\, y\notin T} [p(x, y)\, g_{xy} - p(y, x)\, g_{yx}]\\ &+ \sum_{x\in T,\, y\notin T} [p(y, x)\, h_{xy} - p(x, y)\, h_{yx}]. \end{aligned} \tag{10}$$

Proof of Theorem 1.1. Let $\nu \in \tilde{\mathscr{I}}$. Then $\int \tilde{\Omega}(\sum_{x\in T} f_x)\, d\nu = 0$ for each finite $T \subset \mathbb{Z}$ so that for $T_{[m,n]} = \{x \in \mathbb{Z} : m \leqslant x \leqslant n\}$ we get

$$\begin{aligned}&\sum_{x\in T_{[m,n]},\, y\in \mathbb{Z}} (p(x, y)+p(y, x)) \int f_{yx}\, d\nu\\ &= \sum_{x\in T_{[m,n]},\, y\notin T_{[m,n]}} p(x, y) \int (g_{xy} - h_{yx})\, d\nu\\ &+ \sum_{x\in T_{[m,n]},\, y\notin T_{[m,n]}} p(y, x) \int (h_{xy} - g_{yx})\, d\nu. \end{aligned} \tag{11}$$

Notice that the left-hand side of this equation is increasing in n and $-m$, so that when we take the limit as $n \to \infty$ or as $-m \to \infty$, a limit exists.

Choosing $\epsilon > 0$ we can find N so that for $n > N$:

$$\sum_{x>n+N} \sum_{z<n-x} p(x, x+z)$$
$$\leqslant \sum_{x>n+N} \sum_{z<n-x} |p(x, x+z) - q_1(z)| + \sum_{|z|>N} |z|\, q_1(z) < \frac{\epsilon}{3}$$

$$\sum_{0<x<n} \sum_{z>n-x+N} p(x, x+z)$$
$$\leqslant \sum_{0<x<n} \sum_{z>n-x+N} |p(x, x+z) - q_1(z)| + \sum_{|z|>N} |z|\, q_1(z) < \frac{\epsilon}{3}$$

$$\sum_{x\leqslant 0} \sum_{z>n+N-x} p(x, x+z)$$
$$\leqslant \sum_{x\leqslant 0} \sum_{z>n+N-x} |p(x, x+z) - q_2(z)| + \sum_{|z|>N} |z|\, q_2(z) < \frac{\epsilon}{3}$$

and

$$\sum_{x<-n-N} \sum_{z>-x-n} p(x, x+z)$$
$$\leqslant \sum_{x<-n-N} \sum_{z>-x-n} |p(x, x+z) - q_2(z)| + \sum_{|z|>N} |z|\, q_2(z) < \frac{\epsilon}{3}$$

$$\sum_{-n<x<0} \sum_{z<-x-n-N} p(x, x+z)$$
$$\leqslant \sum_{-n<x<0} \sum_{z<-x-n-N} |p(x, x+z) - q_2(z)| + \sum_{|z|>N} |z|\, q_2(z) < \frac{\epsilon}{3}$$

$$\sum_{x\geqslant 0} \sum_{z<-n-N-x} p(x, x+z)$$
$$\leqslant \sum_{x\geqslant 0} \sum_{z<-n-N-x} |p(x, x+z) - q_1(z)| + \sum_{|z|>N} |z|\, q_1(z) < \frac{\epsilon}{3}.$$

Since the construction of the exclusion process assumes that $\sup_y \sum_x p(x, y)$ is finite (see Chapter VIII in IPS) and since $\int (g_{xy} - h_{yx})\, dv \leqslant 1$, the right-hand side sums in (11) above are absolutely convergent for any fixed n and m.

Now by the inequalities above and by (3) we can pass to the limit in (11) so as to write

$$\lim_{m\to-\infty}\lim_{n\to\infty}\sum_{x\in T_{[m,n]},\,y\in\mathbb{Z}}(p(x,y)+p(y,x))\int f_{yx}\,dv$$

$$=\lim_{n\to\infty}\sum_{x\in T_{[0,n]},\,y>n}\left[q_1(y-x)\int(g_{xy}-h_{yx})\,dv+q_1(x-y)\int(h_{xy}-g_{yx})\,dv\right]$$

$$+\lim_{m\to-\infty}\sum_{x\in T_{[m,0]},\,y<m}\left[q_2(y-x)\int(g_{xy}-h_{yx})\,dv+q_2(x-y)\int(h_{xy}-g_{yx})\,dv\right].$$

The right-hand side above is equal to

$$\lim_{l\to\infty}\frac{1}{l}\sum_{n=1}^{l}\sum_{x\in T_{[0,n]},\,y>n}\left[q_1(y-x)\int(g_{xy}-h_{yx})\,dv+q_1(x-y)\int(h_{xy}-g_{yx})\,dv\right]$$

$$+\lim_{k\to\infty}\frac{1}{k}\sum_{m=-1}^{-k}\sum_{x\in T_{[m,0]},\,y<m}\left[q_2(y-x)\int(g_{xy}-h_{yx})\,dv\right.$$

$$\left.+q_2(x-y)\int(h_{xy}-g_{yx})\,dv\right]. \tag{12}$$

We will devote the next few paragraphs to showing that these limits are in fact equal to zero.

Define the measures v^+ and v^- by choosing a subsequence n_j so that the following limits exist:

$$v^+=\lim_{j\to\infty}\frac{1}{n_j}\sum_{1\leqslant x\leqslant n_j}v_x$$

$$v^-=\lim_{j\to\infty}\frac{1}{|n_{-j}|}\sum_{-1\geqslant x\geqslant n_{-j}}v_x$$

where v_x is the x translate of v. In the partial sums of (12) above, for j large enough each term

$$q_i(y-x)\int(g_{xy}-h_{yx})\,dv$$

appears $|y-x|$ times when $q_i(y-x)>0$, so we can write (12) as

$$\sum_{z\in\mathbb{Z}^+}\left[zq_1(z)\int(g_{oz}-h_{zo})\,dv^+-zq_1(-z)\int(g_{zo}-h_{oz})\,dv^+\right]$$

$$+\sum_{z\in\mathbb{Z}^-}\left[-zq_2(z)\int(g_{oz}-h_{zo})\,dv^-+zq_2(-z)\int(g_{zo}-h_{oz})\,dv^-\right] \tag{13}$$

Now consider two coupled processes with transition rates equal to $q_1(z)$ and $q_2(z)$ respectively. The measures ν^+ and ν^- are translation invariant and are also invariant measures for the coupled process with respect to $q_1(z)$ and $q_2(z)$ respectively. In particular if $\tilde{\Omega}_i$ is the generator for the coupled process of $q_i(z)$, $\tilde{\Omega}$ is the generator for the coupled process of $p(x, y)$, and

$$f_{(A, B)} = \begin{cases} 1 & \text{when } \eta(x) = \xi(y) = 1 \text{ for all } x \text{ in the finite set } A, \\ & \qquad y \text{ in the finite set } B \\ 0 & \text{otherwise} \end{cases}$$

then

$$\begin{aligned} \int \tilde{\Omega}_1 f_{(A, B)}\, d\nu_+ &= \lim_{k \to \infty} \frac{1}{n_k} \sum_{1 \leqslant x \leqslant n_k} \int \tilde{\Omega}_1 f_{(A+x, B+x)}\, d\nu \\ &= \lim_{k \to \infty} \frac{1}{n_k} \sum_{1 \leqslant x \leqslant n_k} \int \tilde{\Omega} f_{(A+x, B+x)}\, d\nu = 0 \end{aligned}$$

where $A + x$ is the x translate A.

By Lemma VIII.3.2 in IPS we have $\int f_{xy}\, d\nu^+ = 0$ for all x, y. We can therefore write ν^+ as $\nu^+ = \lambda \nu_1 + (1 - \lambda)\, \nu_2$ where ν_1 concentrates on $\{(\eta, \xi): \eta < \xi\}$ and ν_2 on $\{(\eta, \xi): \eta \geqslant \xi\}$. Then

$$\begin{aligned} \int (g_{oz} - h_{zo})\, d\nu^+ = \lambda \int (g_{oz} - h_{zo})\, d\nu_1 & \\ = \lambda[\nu_1\{(\eta, \xi): \eta(0) = 1, \eta(z) = 0\} & \\ - \nu_1\{(\eta, \xi): \eta(0) = \xi(0) = 1, \eta(z) = \xi(z) = 0\} & \\ + \nu_1\{(\eta, \xi): \eta(0) = \xi(0) = 1, \eta(z) = \xi(z) = 0\} & \\ - \nu_1\{(\eta, \xi): \xi(0) = 1, \xi(z) = 0\}] & \\ = \lambda[\nu_1\{(\eta, \xi): \eta(0) = 1, \eta(z) = 0\} & \\ - \nu_1\{(\eta, \xi): \xi(0) = 1, \xi(z) = 0\}]. & \end{aligned}$$

Because ν^+ is translation invariant and invariant for the process with rates $q_1(z)$, ν_1 is also since ν_1 and ν_2 are mutually singular and $\nu^+ = \lambda \nu_1 + (1 - \lambda)\, \nu_2$. By Theorem VIII.3.9 in IPS, the marginals of ν_1 are exchangeable, thus the right-hand side above is equal to a constant c^+ as is the expression $\int (g_{zo} - h_{oz})\, d\nu^+$. Similarly we have that $\int (g_{oz} - h_{zo})\, d\nu^-$ and $\int (g_{zo} - h_{oz})\, d\nu^-$ are equal to a constant c^-. Now by the mean zero assumption, we have that expression (13) is equal to 0, but since (12) and (13) are equal, we have in fact that

$$\sum_{y \in T} (p(x, y)+p(y, x)) \int f_{xy}\, dv = 0 \tag{14}$$

for each finite $T \subset \mathbb{Z}$.

By irreducibility, if $\int f_{xy}\, dv > 0$ for some x, y then $\int f_{xy}\, dv > 0$ for all x, y. Choose x_0 and y_0 such that $p(x_0, y_0)+p(y_0, x_0) > 0$. By (14) and the nonnegativity of $\int f_{xy}\, dv$, we must have that $\int f_{x_0 y_0}\, dv = 0$ and thus $\int f_{xy}\, dv = 0$ for all x, y. Therefore $v \in \tilde{\mathscr{I}}$ implies that

$$v\{(\eta, \xi): \eta < \xi \text{ or } \eta \geqslant \xi\} = 1.$$

If $\sum_x \pi(x)/[1+\pi(x)]^2 = \infty$ we can use Theorem 2.1 to pick $\mu \in \mathscr{I}_e$ and $v^c \in \mathscr{I}_e$. On the other hand if $\sum_x \pi(x)/[1+\pi(x)]^2 < \infty$ we can use the analysis in the introduction to pick $\mu \in \mathscr{I}_e$ and $v^{(n)} \in \mathscr{I}_e$. Since $v\{(\eta, \xi): \eta < \xi \text{ or } \eta \geqslant \xi\} = 1$, Proposition VIII.2.13 in IPS tells us there exists a coupling with invariant measure v where v has marginals $\mu \leqslant v^c$ or $\mu \geqslant v^c$ in the first case and marginals $\mu \leqslant v^{(n)}$ or $\mu \geqslant v^{(n)}$ in the second case.

Take first the case where $\sum_x \pi(x)/[1+\pi(x)]^2 = \infty$. Supposing that $\mu \neq v^0 \neq v^\infty$, we have that there exists a c_0 for which $v^{c_1} \leqslant \mu$ for all $c_1 < c_0$ and $\mu \leqslant v^{c_2}$ for all $c_2 > c_0$. By the continuity of the one parameter family of measures $\{v^c\}$ it must be that $\mu = v^{c_0}$.

If $\sum_x \pi(x)/[1+\pi(x)]^2 < \infty$ then we have three cases (i), (ii), and (iii) as given in the introduction. Theorem VIII.2.17 in IPS proves the first two cases so we will consider only (iii). If $\mu \neq v^{(-\infty)} \neq v^{(\infty)}$ then there exists an $n \in \mathbb{Z}$ such that either $\mu = v^{(n)}$ or $v^{(n)} < \mu < v^{(n+1)}$. If the latter is true then μ concentrates on $A = \{\eta: \sum_{x \in T} \eta(x) < \infty, \sum_{x \notin T} [1-\eta(x)] < \infty\}$ for some $T \subset S$ which means that it must be a mixture of stationary distributions for the Markov chains on A_n as described in the introduction. But $\mu \in \mathscr{I}_e$ so it must in fact be equal to some $v^{(n)}$ completing the proof. ▮

We include in this section two more results which have proofs similar to that of Theorem 1.1. We first need the following definition: given transition probabilities $p(x, y)$ define the boundary of a set $\mathscr{T}$ to be

$$\partial\mathscr{T} = \{x \notin \mathscr{T} : p(x, y) > 0 \text{ for some } y \in \mathscr{T}\}.$$

Proposition 3.1. Let $\mathscr{S}$ be irreducible with respect to $p(x, y)$ and suppose that $\sum_x \pi(x)/[1+\pi(x)]^2 = \infty$. If there exists a sequence of increasing finite sets $\mathscr{T}_n$ such that $\bigcup \mathscr{T}_n = \mathscr{S}$ and either $\lim_{n \to \infty} \sum_{x \in \partial\mathscr{T}_n} \pi(x) = 0$ or $\lim_{n \to \infty} \sum_{x \in \partial\mathscr{T}_n} 1/\pi(x) = 0$, then $\mathscr{I}_e = \{v^c: 0 \leqslant c \leqslant \infty\}$.

Proof. Choose $\mu \in \mathscr{I}_e$. If $\lim_{n\to\infty} \sum_{x\in\partial\mathscr{T}_n} \pi(x) = 0$ then couple η_t with ξ_t so that they have the measures μ and ν^c respectively. If $\lim_{n\to\infty} \sum_{x\in\partial\mathscr{T}_n} 1/\pi(x) = 0$ then couple them vice versa. We will prove the case in which $\lim_{n\to\infty} \sum_{x\in\partial\mathscr{T}_n} \pi(x) = 0$. The other case follows similarly.

By (10),

$$\sum_{x\in\mathscr{T}_n,\, y\in\mathscr{S}} [p(x, y)+p(y, x)] \int f_{yx}\, d\nu$$

$$= \sum_{x\in\mathscr{T}_n,\, y\notin\mathscr{T}_n} p(x, y) \int (g_{xy} - h_{yx})\, d\nu + \sum_{x\in\mathscr{T}_n,\, y\notin\mathscr{T}_n} p(y, x) \int (h_{xy} - g_{yx})\, d\nu.$$

Just as in the above proof, the left-hand side of this equation is increasing in n so that a limit exists as $n\to\infty$. The right-hand side above goes to 0 as $n\to\infty$ since

$$\sum_{x\in\mathscr{T}_n,\, y\notin\mathscr{T}_n} p(x, y) \int (g_{xy} - h_{yx})\, d\nu + \sum_{x\in\mathscr{T}_n,\, y\notin\mathscr{T}_n} p(y, x) \int (h_{xy} - g_{yx})\, d\nu$$

$$\leqslant \sum_{x\in\mathscr{T}_n,\, y\notin\mathscr{T}_n} p(x, y) \int f_y\, d\nu + \sum_{x\in\mathscr{T}_n,\, y\notin\mathscr{T}_n} p(y, x) \int f_y\, d\nu$$

$$\leqslant C \sum_{y\in\partial\mathscr{T}_n} \int f_y\, d\nu + \sum_{y\in\partial T_n} \int f_y\, d\nu \leqslant C \sum_{y\in\partial\mathscr{T}_n} \pi(y) + \sum_{y\in\partial T_n} \pi(y).$$

Here $C = \sup_y \sum_x p(x, y)$ which is finite by the assumptions in the introduction.

Irreducibility now gives us $\int f_{xy}\, d\nu = 0$ for all x, y. The rest of the proof just follows that of Theorem 1.1. ∎

Note that if we change the hypothesis $\sum_x \pi(x)/[1+\pi(x)]^2 = \infty$ to $\sum_x \pi(x)/[1+\pi(x)]^2 < \infty$ then Theorem VIII.2.17 in IPS says that $\mathscr{I}_e = \{\nu^{(n)}: 0 \leqslant n \leqslant \infty\}$.

Corollary 3.2. If in Theorem 1.1 we replaced condition (3) with the condition that $p(x, y)$ has finite range, $\lim_{x\to+\infty} p(x, x+z) = q_1(z)$, and $\lim_{x\to-\infty} \pi(x)$ equals 0 or ∞ (or alternatively $\lim_{x\to-\infty} p(x, x+z) = q_2(z)$, and $\lim_{x\to+\infty} \pi(x)$ equals 0 or ∞) then the result still holds.

Proof. Replace expression (12) in the proof of Theorem 1.1 with

$$\lim_{k\to\infty} \frac{1}{k} \sum_{n=1}^{k} \sum_{x\in T_{[0,n]},\, y>n} \left[q_1(y-x) \int (g_{xy} - h_{yx})\, d\nu + q_1(x-y) \int (h_{xy} - g_{yx})\, d\nu \right]$$

$$+ \lim_{m\to-\infty} \sum_{x\in T_{[m,0]},\, y<m} \left[p(x, y) \int (g_{xy} - h_{yx})\, d\nu + p(y, x) \int (h_{xy} - g_{yx})\, d\nu \right].$$

The proofs of Theorem 1.1 and Proposition 3.1 imply that this expression is 0. The rest is proven above. ▮

Before moving on to the next section let us discuss what the above results tell us in the case where $p(x, y)$ has finite range on $\mathbb{Z}$. Proposition 3.1 together with Theorem VIII.2.17 in IPS says that if $\lim_{|x| \to \infty} \pi(x)$ equals 0 or ∞ then the reversible measures are the only invariant measures. If the limits $\lim_{x \to \infty} \pi(x)$ and $\lim_{x \to -\infty} \pi(x)$ exist and one of them is nonzero and finite, then the combination of Theorem 1.1 and Corollary 3.2 imply that the only invariant measures are the reversible ones. All together we have the following: if $\pi(x)$ exists and has limits in both directions for the finite range exclusion process on $\mathbb{Z}$, then unless the limit is 0 in one direction and ∞ in the other direction, the only invariant measures are the reversible ones. Of course, as seen in an example in the introduction, it is also possible to have $\lim_{x \to \infty} p(x, x+z) = q_1(z)$ and $\lim_{x \to -\infty} p(x, x+z) = q_2(z)$ as given in Theorem 1.1 and at the same time have the limit of $\pi(x)$ to be 0 in one direction, ∞ in the other. In those cases Theorem 1.1 rules out nonreversible invariant measures. A similar comment can be made for Corollary 3.2. We remind the reader, however, that if the transition probabilities are translation invariant with a drift so that the limit of $\pi(x)$ is 0 in one direction and ∞ in the other direction, then Liggett[7] tells us that $\{\nu_\rho : 0 \leqslant \rho \leqslant 1\}$ is a class of nonreversible invariant measures.

4. THE NEAREST-NEIGHBOR PROCESS ON $\mathbb{Z}$

We now restrict our attention to the nearest-neighbor case. More specifically, assume throughout this section that we are dealing with the irreducible nearest-neighbor exclusion process on $\mathbb{Z}$ ($p(x, y) = 0$ if and only if $|x-y| > 1$). In this case, a reversible $\pi(x)$ always exists so we need not make this assumption. Similar to the discussion at the end of the last section, we will show that if $\inf_{|x-y|=1} p(x, y) > 0$ then the only possible nonreversible measures are in the case where the limit of $\pi(x)$ is 0 in one direction and ∞ in the other direction.

In order to prove the next two propositions we need the following lemma which appears in a slightly different form as Corollary 5.2 in Liggett:[7]

Lemma 4.1 (Liggett). If $\inf_{|x-y|=1} p(x, y) > 0$ and $\nu \in \tilde{\mathscr{I}}_e$, then exactly one of the following holds:

(a) $\nu\{(\eta, \xi) : \eta = \xi\} = 1$,

(b) $\nu\{(\eta, \xi) : \eta \leqslant \xi, \eta \neq \xi\} = 1$,

(c) $\nu\{(\eta, \xi): \eta \geqslant \xi, \eta \neq \xi\} = 1$,

(d) $\nu(B) = 1$,

(e) $\nu\{(\eta, \xi): (\xi, \eta) \in B\} = 1$,

where $B = \{(\eta, \xi): \exists x \in \mathbb{Z}$ such that $\eta(y) \leqslant \xi(y)$ for all $y < x, \eta(y) < \xi(y)$ for some $y < x, \eta(z) \geqslant \xi(z)$ for all $z \geqslant x, \eta(z) > \xi(z)$ for some $z \geqslant x\}$.

Proposition 4.2. If $\inf_{|x-y|=1} p(x, y) > 0$ and $\pi(x)$ has some finite, nonzero limit point as x goes to ∞ and some finite, nonzero limit point as x goes to $-\infty$, then $\mathcal{I}_e = \{\nu^c: 0 \leqslant c \leqslant \infty\}$.

Proof. The assumptions imply that $\sum_x \pi(x)/[1+\pi(x)]^2 = \infty$ so Theorem 2.1 tells us $\mathcal{I}_e \supset \{\nu^c: 0 \leqslant c \leqslant \infty\}$. We will show the reverse containment.

Choose a sequence $\{n_k\}$ extending in both directions so that finite, nonzero limits of $\pi(n_k)$ exist. For a measure μ on $\{0, 1\}^{\mathbb{Z}}$ the set of limit points L_+ of $\{\mu\{\xi(n_k) = 1\}, k > 0\}$ satisfies one of the following properties:

(i) $L_+ = \{1\}$ or $L_+ = \{0\}$.

(ii) $L_+ = \{1, 0\}$.

(iii) L_+ contains some limit point between 0 and 1.

The same is true for the set of limit points L_- of $\{\mu\{\xi(n_k) = 1\}, k < 0\}$.

Now suppose we couple ν^c with another extremal invariant measure μ_e, the two measures corresponding to the processes η_t and ξ_t respectively. Since Theorem 2.1 tells us that ν^c is extremal, Section VIII.2 in IPS implies there exists a coupling measure such that $\nu \in \tilde{\mathcal{I}}_e$.

If μ_e satisfies condition (i) for both L_+ and L_- then there are two possibilities: either $L_+ = L_-$ or $L_+ \neq L_-$. Suppose first that $L_+ = L_- = \{1\}$ for μ_e. If in this case we have that $\mu_e\{\xi(z) = 1\} < 1$ for some z then we can choose $c < \infty$ large enough so that $\nu^c\{\eta(z) = 1\} > \mu_e\{\xi(z) = 1\}$. But this contradicts the assumption that $\nu^c\{\eta(n_k) = 1\} = c\pi(n_k)/[1+c\pi(n_k)]$ has limits less than 1 for k going to ∞ and $-\infty$. To see this suppose the coupling measure satisfies $\nu(B) = 1$ as defined in Lemma 4.1. Given

$$0 < \epsilon < 1 - \lim_{k \to \infty} c\pi(n_k)/[1+c\pi(n_k)] \tag{15}$$

we can choose K large enough so that

$$1-\epsilon < \nu\{(\eta, \xi): \exists x < K \text{ such that } \eta(y) \leqslant \xi(y)\ \forall y < x,\ \eta(y) < \xi(y)$$
$$\text{for some } y < x,\ \eta(z) \geqslant \xi(z)\ \forall z \geqslant x,\ \eta(z) > \xi(z) \text{ for some } z \geqslant x\}.$$

This, however, contradicts $L_+ = 1$. Similarly we cannot have that $\nu\{(\eta, \xi): (\xi, \eta) \in B\} = 1$. So Lemma 4.1 tells us that $\eta \leqslant \xi$ which contradicts $\nu^c\{\eta(z) = 1\} > \mu_e\{\xi(z) = 1\}$. It must be that $\mu_e = \nu^\infty$. A similar argument shows that if $L_+ = L_- = \{0\}$ for μ_e then $\mu_e = \nu^0$.

Consider the second case where $L_- \neq L_+$; without loss of generality we will assume that $L_- = \{0\}$.

We claim that given $\epsilon > 0$, we can find n such that $\mu_e\{\xi(n) = 0\} < \epsilon$ and $\mu_e\{\xi(n+1) = 0\} < \epsilon$. To see this suppose that for some $\epsilon > 0$ there exists no n for which this is true. Then since $L_+ = \{1\}$, there are infinitely many $x > 0$ for which $\mu_e\{\xi(x) = 0\} < \epsilon/4$ and infinitely many $y > 0$ for which $\mu_e\{\xi(y) = 0\} \geqslant \epsilon$. Choosing ν^c so that $\lim_{k\to\infty} c\pi(n_k)/[1 + c\pi(n_k)] = 1 - \epsilon/2$ gives us a contradiction to Lemma 4.1 and thus proves the claim.

Given the same $\epsilon > 0$ we can choose $m < n$ so that $\mu_e\{\xi(m-1) = 1\} < \epsilon$. Since we have that $\nu \in \tilde{\mathscr{I}}_e$ then $\int \tilde{\Omega}(\sum_{x\in T} f_x)\, d\nu = 0$ for each finite $T \subset \mathbb{Z}$. By (10),

$$\sum_{m \leqslant x \leqslant n,\, y\in\mathbb{Z}} (p(x, y) + p(y, x)) \int f_{yx}\, d\nu$$

$$= \sum_{x = m \text{ or } n,\, y = m-1 \text{ or } n+1} \left[p(x, y) \int (g_{xy} - h_{yx})\, d\nu + p(y, x) \int (h_{xy} - g_{yx})\, d\nu \right] \tag{16}$$

which is increasing in n and $-m$.

Due to our choice of m and n above, $\int h_{n,n+1}\, d\nu < \epsilon$ and $\mu_e\{\xi(m-1) = 1\} < \epsilon$; moreover $P(A) - P(A \cap B \cap C) \leqslant P(B^c) + P(C^c)$ implies that $\nu^c\{\eta(n+1) = 1, \eta(n) = 0\} - \int g_{n+1,n}\, d\nu < 2\epsilon$ so that

$$\sum_{m \leqslant x \leqslant n,\, y\in\mathbb{Z}} (p(x, y) + p(y, x)) \int f_{yx}\, d\nu$$

$$< p(n, n+1) \int g_{n,n+1}\, d\nu - p(n+1, n) \int g_{n+1,n}\, d\nu + 3\epsilon$$

$$< p(n, n+1)\, \nu^c\{\eta(n) = 1, \eta(n+1) = 0\}$$
$$- p(n+1, n)\, \nu^c\{\eta(n) = 0, \eta(n+1) = 1\} + 5\epsilon.$$

By the reversibility of ν^c

$$p(n, n+1)\, \nu^c\{\eta(n) = 1, \eta(n+1) = 0\} = p(n+1, n)\, \nu^c\{\eta(n) = 0, \eta(n+1) = 1\}$$

so Eq. (16) is in fact equal to 0. Since we have assumed here that $L_- = \{0\}$ and $L_+ = \{1\}$ for μ_e then choosing $0 < c < \infty$ gives us a contradiction.

Suppose μ_e satisfies condition (ii) for either L_+ or L_- so that either $L_+ = \{0, 1\}$ or $L_- = \{0, 1\}$. Choose ν^c with $0 < c < \infty$. Again we contradict Lemma 4.1.

Combining all the above arguments we have that either $\mu_e = \nu^0$, $\mu_e = \nu^\infty$, or μ_e satisfies (iii) in some direction. Assuming the latter we can, without loss of generality, choose $0 < c_0 < \infty$ so that

$$\lim_{k\to\infty} c_0\pi(n_k)/[1+c_0\pi(n_k)] = \lim_{l\to\infty} \mu_e\{\xi(n_{k_l}) = 1\}.$$

For all $c > c_0$,

$$\lim_{k\to\infty} c\pi(n_k)/[1+c\pi(n_k)] > \lim_{l\to\infty} \mu_e\{\xi(n_{k_l}) = 1\}.$$

By Lemma 4.1 either $\mu_e \leqslant \nu^c$ or $\nu(B) = 1$ where B is defined in the lemma. Similarly, for all $c < c_0$, either $\mu_e \geqslant \nu^c$ or $\nu\{(\eta, \xi): (\xi, \eta) \in B\} = 1$. Combining these two arguments gives $\nu^{c_1} \leqslant \mu_e \leqslant \nu^{c_2}$ for all $c_1 < c_0 < c_2$. By the continuity of the one parameter family of measures ν^c, $\mu_e = \nu^{c_0}$. ∎

Proposition 4.3. If $\inf_{|x-y|=1} p(x, y) > 0$, $\lim_{x\to\infty} \pi(x) = \infty$, and $\pi(x)$ has a finite, nonzero limit point as x goes to $-\infty$, then $\mathscr{I}_e = \{\nu^c: 0 \leqslant c \leqslant \infty\}$.

Proof. Again, by Theorem 2.1 we need only show that $\mathscr{I}_e \subset \{\nu^c: 0 \leqslant c \leqslant \infty\}$.

We argue first that without loss of generality we can assume the limit points of $\{\pi(x), x < 0\}$ are all finite. Assume to the contrary that ∞ is a limit point. For any $R > 0$ we can find $x < -R$ such that $\min(\pi(x), \pi(x+1)) > R$ since $\inf_{|x-y|=1} p(x, y) > 0$. The conditions of Proposition 3.1 are then satisfied so that $\mathscr{I}_e = \{\nu^c: 0 \leqslant c \leqslant \infty\}$ holds.

Couple ν^c with another extremal invariant measure μ_e, the two measures corresponding to the processes η_t and ξ_t respectively. As argued above there exists a coupling measure such that $\nu \in \tilde{\mathscr{I}}_e$.

Let L^- be the the set of limit points of $\{\mu_e\{\xi(x) = 1\}, x < 0\}$. Note that L^- is slightly different from L_- described in Proposition 4.2 in that L_- is the set of limit points for a subset of $\{\mu_e\{\xi(x) = 1\}, x < 0\}$. L^- satisfies one of the following properties:

(i) L^- contains some limit point between 0 and 1.

(ii) $L^- = \{1, 0\}$.

(iii) $L^- = \{1\}$.

(iv) $L^- = \{0\}$.

The same is true for the set L^+ of limit points $\{\mu_e\{\xi(x) = 1\}, x > 0\}$.

Suppose L^- satisfies (i). Choose a sequence $x_n \to -\infty$ so that $0 < \lim_{n\to\infty} \mu_e\{\xi(x_n)=1\} < 1$ exists. Since we can assume that the limit points of $\{\pi(x), x<0\}$ are all finite, there exists a subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty} \pi(x_{n_k}) < \infty$ exists.

Consider the two cases where $\lim_{k\to\infty} \pi(x_{n_k}) = 0$ and where $\lim_{k\to\infty} \pi(x_{n_k}) > 0$. Assume the latter case first. Choose $0 < c_0 < \infty$ so that

$$\lim_{k\to\infty} c_0\pi(x_{n_k})/[1+c_0\pi(x_{n_k})] = \lim_{n\to\infty} \mu_e\{\xi(x_n)=1\}.$$

For all $c > c_0$,

$$\lim_{k\to\infty} c\pi(x_{n_k})/[1+c\pi(x_{n_k})] > \lim_{n\to\infty} \mu_e\{\xi(x_n)=1\}.$$

Using the argument at the end of Proposition 4.2, we have that for all $c_1 < c_0 < c_2$, $v^{c_1} \leqslant \mu_e \leqslant v^{c_2}$. Consequently, it must be that $\mu_e = v^{c_0}$.

Now assume that $\lim_{k\to\infty} \pi(x_{n_k}) = 0$ so that for all $0 < c < \infty$ the coupling satisfies either $v^c \leqslant \mu_e$ or $v\{B\} = 1$ where B is given in Lemma 4.1. If $v^c \leqslant \mu_e$ for all $0 < c < \infty$ then $\mu_e = v^\infty$, a contradiction to L^- satisfying (i). So it must be that $v\{B\} = 1$.

We claim that for any $r < 1$ there exists $m < 0$ such that $\mu_e\{\xi(m) = 1\} > r$ and $\mu_e\{\xi(m-1) = 1\} > r$. By the hypothesis of the theorem we can choose a sequence $\{x_l\}$ going to $-\infty$ so that $0 < \lim_{l\to\infty} \pi(x_l) < \infty$ exists. If $\inf_{|x-y|=1} p(x, y) > p$ then choose c so that

$$\lim_{l\to\infty} \frac{cp\pi(x_l)}{1+cp\pi(x_l)} > r + \frac{1-r}{2}.$$

Since $\pi(x_l - 1) > p\pi(x_l)$, it follows that the set of limit points of $\{\frac{c\pi(x_l-1)}{1+c\pi(x_l-1)}, l > 0\}$ is bounded below by $r + \frac{1-r}{2}$. Now since $v\{B\} = 1$ there exists a K such that $l > K$ implies $\mu_e\{\xi(x_l) = 1\} > r$ and $\mu_e\{\xi(x_l - 1) = 1\} > r$ which proves the claim.

Since we have that $v \in \tilde{\mathscr{I}}_e$ then $\int \tilde{\Omega}(\sum_{x\in T} f_x)\, dv = 0$ for each finite $T \subset \mathbb{Z}$. By (10),

$$\sum_{m\leqslant x\leqslant n,\, y\in\mathbb{Z}} (p(x, y)+p(y, x)) \int f_{yx}\, dv$$

$$= \sum_{x=m \text{ or } n,\, y=m-1 \text{ or } n+1} \left[p(x, y) \int (g_{xy} - h_{yx})\, dv + p(y, x) \int (h_{xy} - g_{yx})\, dv \right]$$

which is increasing in n and $-m$.

Using the claim above along with the fact that $\lim_{x\to\infty} \pi(x) = \infty$, we can argue just as we argued in the case where $L_- \neq L_+$ of (i) in Proposition 4.2, to get

$$\sum_{m \leqslant x \leqslant n,\, y \in \mathbb{Z}} (p(x, y)+p(y, x)) \int f_{yx}\, dv$$

$$< p(m, m-1) \int g_{m, m-1}\, dv - p(m-1, m) \int g_{m-1, m}\, dv + 3\epsilon$$

$$< p(m, m-1)\, v^c\{\eta(m)=1, \eta(m-1)=0\}$$

$$-p(m-1, m)\, v^c\{\eta(m)=0, \eta(m-1)=1\}+5\epsilon.$$

By the reversibility of v^c the left-hand side must be 0, but this contradicts $v\{B\}=1$.

Suppose L^- satisfies condition (ii). Choosing v^c with $0<c<\infty$ gives us a contradiction to Lemma 4.1.

If L^- satisfies condition (iii) then we will handle the two cases (a) $L^+=\{1\}$ and (b) $L^+ \neq \{1\}$. Considering case (a) if we switch the coupling so that μ_e corresponds to η_t then we have that the right-hand side of the following inequality goes to 0:

$$\sum_{|x|=n,\, |y|=n+1} p(x, y) \int (g_{xy}-h_{yx})\, dv + \sum_{|x|=n,\, |y|=n+1} p(y, x) \int (h_{xy}-g_{yx})\, dv$$

$$\leqslant \sum_{|x|=n,\, |y|=n+1} (p(x, y)+p(y, x)) \int f_y\, dv. \tag{17}$$

By (10) and by irreducibility we get $\int f_{xy}\, dv=0$ for all x, y. The measure μ_e must lie stochastically above all v^c for all finite c and must therefore be equal to v^∞.

If (b) holds then we refer the reader to the argument given above in the case where L^- satisfies (i) and $\lim_{k \to \infty} \pi(x_{n_k})=0$.

Finally suppose that (iv) holds so that $L^-=\{0\}$. If L^+ satisfies (i) or (ii) then by Lemma 4.1, $\mu_e \leqslant v^c$ for all $c>0$ so that $\mu_e=v^0$, a contradiction. If L^+ satisfies (iv) then similarly $\mu_e=v^0$. Let L^+ satisfy (iii) so that $L^+=\{1\}$. For a given z choose c small enough so that $v^c\{\eta(z)=1\}<\mu_e\{\xi(z)=1\}$. We thus have that $v\{(\eta, \xi)\colon (\xi, \eta) \in B\}=1$ as given in Lemma 4.1. But by (10) and (17), for a given $\epsilon>0$ we can find $-m$ and n large enough so that

$$\sum_{m \leqslant x \leqslant n,\, y \in \mathbb{Z}} (p(x, y)+p(y, x)) \int f_{yx}\, dv < \epsilon$$

which of course contradicts $v\{(\eta, \xi)\colon (\xi, \eta) \in B\}=1$. ∎

Proof of Theorem 1.2. Note first that since $\inf_{|x-y|=1} p(x, y) > 0$ then it cannot be that $\mathscr{L}^-$ or $\mathscr{L}^+$ is equal to $\{0, \infty\}$. In light of this fact, if either $\mathscr{L}^-$ or $\mathscr{L}^+$ contains a finite, nonzero point then Proposition 4.2 and analogs of Proposition 4.3 imply there are no nonreversible measures. If $\mathscr{L}^+ = \mathscr{L}^- = \{0\}$ or $\mathscr{L}^+ = \mathscr{L}^- = \{\infty\}$ then Proposition 3.1 implies there are no nonreversible measures. ∎

5. A RESULT CONCERNING DOMAINS OF ATTRACTION

Theorem 5.1. Let $\sum_x \pi(x)/[1+\pi(x)]^2 = \infty$ and let ω be a probability measure on $[0, \infty]$. Also, assume that ν_c is a family of invariant measures indexed by $c \geqslant 0$ each of which is in $\mathscr{I}_e$. Suppose $\{\mu_c\}$ is a family of probability measures on $\{0, 1\}^{\mathscr{S}}$ such that for each $0 \leqslant c \leqslant \infty$, μ_c is absolutely continuous with respect to ν_c. If

$$\mu = \int_0^\infty \mu_c \omega(dc) \qquad \text{and} \qquad \nu = \int_0^\infty \nu_c \omega(dc) \tag{18}$$

then $\lim_{T\to\infty} \frac{1}{T} \int_0^T \mu S(t)\, dt$ exists and is equal to ν.

Proof. For a fixed c we first prove that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mu_c S(t)\, dt = \nu_c. \tag{19}$$

Let $\mathscr{P}$ be the set of all measures. By the compactness of $\mathscr{P}$ we can choose a sequence of times such that

$$\lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} \mu_c S(t)\, dt \tag{20}$$

converges in distribution to some measure λ. Pick a continuous (and therefore bounded) function f on $\{0, 1\}^{\mathscr{S}}$ with $\|f\| \leqslant 1$ and let g be the Radon–Nikodym derivative of μ_c with respect to ν_c. Given $\epsilon > 0$ we have that for n large enough

$$\left| \frac{1}{t_n} \int_0^{t_n} \int (S(t)\, f)\, g\, d\nu_c\, dt - \int f\, d\lambda \right| < \epsilon/3.$$

We can choose a simple function

$$\hat{g} = \sum_{k=1}^{N} c_k 1_{E_k}$$

approximating g such that $\bigcup_k E_k = \{0, 1\}^{\mathscr{S}}$, $\hat{g} \geqslant 0$, $\int \hat{g}\, d\nu_c = 1$, and $\int |g - \hat{g}|\, d\nu_c < \epsilon/3$. Since $\|S(t)\, f\| \leqslant \|f\| \leqslant 1$ this gives us

$$\left| \frac{1}{t_n} \int_0^{t_n} \int (S(t)\, f)\, g\, d\nu_c\, dt - \frac{1}{t_n} \int_0^{t_n} \int (S(t)\, f)\, \hat{g}\, d\nu_c\, dt \right| \leqslant \int |g - \hat{g}|\, d\nu_c < \epsilon/3.$$

Without loss of generality we can henceforth assume that $\nu_c(E_k) > 0$ for each k. Define the measure μ_k concentrating on E_k by letting

$$\mu_k(A) = \frac{\nu_c(A \cap E_k)}{\nu_c(E_k)}$$

If we think of $\hat{g}$ as the Radon–Nikodym derivative of some measure λ_ϵ with respect to ν_c then we can write

$$\sum_{k=1}^{N} \nu_c(E_k)\, \mu_k = \nu_c \qquad \text{and} \qquad \sum_{k=1}^{N} c_k \nu_c(E_k)\, \mu_k = \lambda_\epsilon.$$

We can now find a subsequence $\{t_{n_l}\}$ such that the following limits exist for each k:

$$\lim_{l \to \infty} \frac{1}{t_{n_l}} \int_0^{t_{n_l}} \mu_k S(t)\, dt = \nu_k.$$

Moreover, Proposition I.1.8 in IPS tells us $\nu_k \in \mathscr{I}$. Since ν_c is extremal invariant and since $\sum_{k \geqslant 1} \nu_c(E_k)\, \nu_k = \nu_c$, it must be that $\nu_k = \nu_c$ for each k. This then yields

$$\sum_{k=1}^{N} c_k \nu_c(E_k)\, \nu_k = \lim_{l \to \infty} \frac{1}{t_{n_l}} \int_0^{t_{n_l}} \lambda_\epsilon S(t)\, dt = \nu_c$$

which gives us

$$\left| \frac{1}{t_{n_l}} \int_0^{t_{n_l}} \int (S(t)\, f)\, \hat{g}\, d\nu_c\, dt - \int f\, d\nu_c \right| < \epsilon/3$$

for l large enough.

Combining the three inequalities we have

$$\left| \int f\, d\lambda - \int f\, d\nu_c \right| < \epsilon.$$

But $\epsilon > 0$ is arbitrary so it must be that $\int f\, d\lambda = \int f\, d\nu_c$ for each continuous f with $\|f\| \leqslant 1$ which implies that (20) is equal to ν_c. Now let M_n be the closure of the set of measures

$$\left\{ \frac{1}{T} \int_0^T \mu_c S(t)\, dt : T \geqslant n \right\}.$$

Using the compactness of $\mathscr{P}$ along with the fact that $\{t_n\}$ is an arbitrary sequence of times causing convergence in (20), we have that $\bigcap_{n \in \mathbb{N}} M_n = \nu_c$ proving (19).

To finish the proof note that since $\|S(t)\, f\| \leqslant \|f\|$, we can use the Dominated Convergence Theorem together with Fubini's Theorem to show that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \int_0^\infty \int S(t)\, f\, d\mu_c\, \omega(dc)\, dt = \int f\, d\nu. \quad \blacksquare$$

For the following corollary let ν_α be the product measure with marginals $0 < \nu_\alpha\{\eta \colon \eta(x) = 1\} = \alpha(x) < 1$ for $\alpha(x)$ a function on $\mathscr{S}$.

Corollary 5.2. Suppose $\sum_x \pi(x)/[1+\pi(x)]^2 = \infty$. If $\sum_x |\alpha(x) - \frac{c\pi(x)}{1+c\pi(x)}| < \infty$ then

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \nu_\alpha S(t)\, dt = \nu^c. \tag{21}$$

Proof. Let $\beta(x) = \frac{c\pi(x)}{1+c\pi(x)}$, $m_x = \min[\alpha(x), \beta(x)]$, and $M_x = \max[\alpha(x), \beta(x)]$. We then have

$$\begin{aligned} 1 - |\alpha(x) - \beta(x)| &= 1 - M_x + m_x \\ &= [(1-M_x)(1-M_x)]^{1/2} + (m_x m_x)^{1/2} \\ &\leqslant [(1-M_x)(1-m_x)]^{1/2} + (m_x M_x)^{1/2} \\ &= [(1-\alpha(x))(1-\beta(x))]^{1/2} + (\alpha(x)\, \beta(x))^{1/2}. \end{aligned}$$

Since $\sum_x |\alpha(x) - \beta(x)| < \infty$ then

$$\prod_x \{(\alpha(x)\, \beta(x))^{1/2} + [(1-\alpha(x))(1-\beta(x))]^{1/2}\} \geqslant \prod_x \{1 - |\alpha(x) - \beta(x)|\} > 0.$$

An application of Kakutani's Dichotomy tells us that ν_α is absolutely continuous with respect to ν^c which completes the proof. $\blacksquare$

We remark here that if $\alpha(x)$ and $\beta(x)$ are both bounded away from 0 and 1 then Kakutani's Dichotomy tells us that $\sum_x [\alpha(x)-\beta(x)]^2 < \infty$ is a necessary and sufficient condition for ν_α to be absolutely continuous with respect to ν^c (e.g., p. 245 of ref. 4).

ACKNOWLEDGMENTS

The author thanks his advisor, Thomas M. Liggett, for motivating this paper and for the many discussions that led to the writing of it. Research supported in part by NSF Grant DMS-00-70465.

REFERENCES

1. E. D. Andjel, Convergence to a nonextremal equilibrium measure in the exclusion process, *Probab. Theory Related Fields* **73**:127–134 (1986).
2. E. D. Andjel, M. D. Bramson, and T. M. Liggett, Shocks in the asymmetric exclusion process, *Probab. Theory Related Fields* **78**:231–247 (1988).
3. M. D. Bramson, T. M. Liggett, and T. S. Mountford, Characterization of stationary measures for one-dimensional exclusion processes, *Ann. Probab.* **30**:1539–1575 (2002).
4. R. Durrett, *Probability: Theory and Examples*, 2nd ed. (Duxbury Press, Belmont, CA, 1996).
5. S. A. Janowski and J. L. Lebowitz, Exact results for the asymmetric simple exclusion with a blockage, *J. Statist. Phys.* **77**:35–51 (1994).
6. C. Kipnis and S. R. S. Varadhan, Central limit theorem for additive functionals of reversible Markov processes, *Comm. Math. Phys.* **104**:1–19 (1986).
7. T. M. Liggett, Coupling the simple exclusion process, *Ann. Probab.* **4**:339–356 (1976).
8. T. M. Liggett, *Interacting Particle Systems* (Springer-Verlag, New York, 1985).
9. T. M. Liggett, *Stochastic Interacting Systems: Contact, Voter, and Exclusion Processes* (Springer-Verlag, Berlin/Heidelberg, 1999).
10. S. Sethuraman, On extremal measures for conservative particle systems, *Ann. Inst. H. Poincaré* **37**:139–154 (2001).